Review exercise 2

1 Changing the units:

diameter = 7 cm ⇒ radius (r) = 3.5 cm = 0.035 m

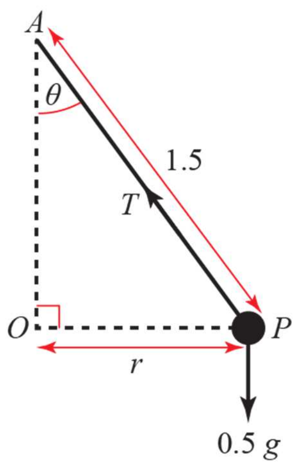
angular speed (ω) = 1000 revolutions per minute $= \dfrac{1000 \times 2\pi}{60}$ radians per second

Using $v = r\omega$ gives:

$$v = 0.035 \times \frac{1000 \times 2\pi}{60} = 3.67\,\text{m s}^{-1} \text{ (3 s.f.)}$$

So the speed is $3.67\,\text{m s}^{-1}$ (3 s.f.)

2 a Let the tension in the string be T and let the string make an angle θ with the vertical.



Let $OP = r$, then $r = 1.5\sin\theta$

Acceleration towards the centre of the circle = $r\omega^2 = 1.5\sin\theta \times 2.7^2$

Force towards the centre of the circle = $T\sin\theta$

So using $F = ma$ gives:

$T\sin\theta = 0.5 \times 1.5\sin\theta \times 2.7^2$

$\Rightarrow T = 0.5 \times 1.5 \times 2.7^2 = 5.4675 = 5.5\,\text{N}$ (2 s.f.)

b Resolving the forces vertically R(↑) and substituting for T gives:

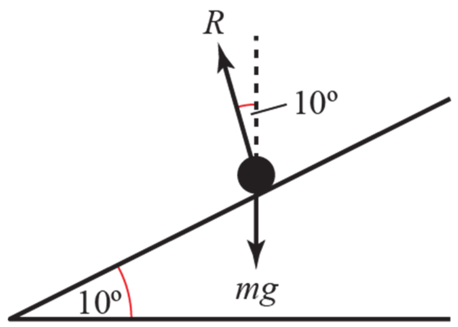
$T\cos\theta - 0.5g = 0$

$\Rightarrow T\cos\theta = 0.5g$

$\Rightarrow \cos\theta = \dfrac{0.5g}{5.4675} = 0.8962$

So $\theta = \cos^{-1} 0.8962 = 26°$ (to the nearest degree)

3 The forces acting on the car are its weight mg and the normal reaction R.



$R(\uparrow): R\cos 10° - mg = 0$

$\Rightarrow R = \dfrac{mg}{\cos 10°}$

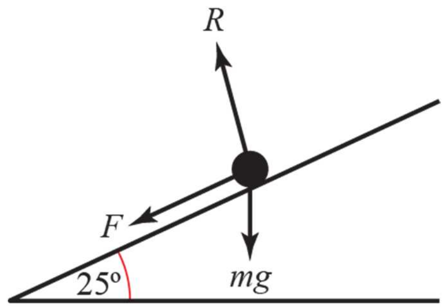
Using $F = ma$ horizontally gives:

$R\sin 10° = \dfrac{mv^2}{r} = \dfrac{m \times 18^2}{r}$

$\Rightarrow \dfrac{mg}{\cos 10°} \times \sin 10° = \dfrac{m \times 18^2}{r}$ substituting for R

$\Rightarrow r = \dfrac{18^2}{g\tan 10°} = 187.4995\ldots = 190\text{ m (2 s.f.)}$

4 The forces acting on the cyclist and bicycle are their weight mg, the normal reaction R and the friction acting down the slope.



$\text{R}(\uparrow)$: $R\cos 25° - F\sin 25° - mg = 0$

If μ is the coefficient of friction between the cycle's tyres and the track, then the maximum friction for which the tyres do not slip is $F = \mu R = 0.6R$. Substituting for F gives:

$R\cos 25° - 0.6R\sin 25° - mg = 0$

$\Rightarrow R = \dfrac{mg}{\cos 25° - 0.6\sin 25°}$ **(1)**

$\text{R}(\leftarrow)$: $R\sin 25° + F\cos 25° = \dfrac{mv^2}{r}$ using $F = ma$ and $a = \dfrac{v^2}{r}$

$\Rightarrow R\sin 25° + 0.6R\cos 25° = \dfrac{mv^2}{40}$ as $r = 40$ and $F = 0.6R$ at maximum speed

$\Rightarrow R = \dfrac{mv^2}{40(\sin 25° + 0.6\cos 25°)}$ **(2)**

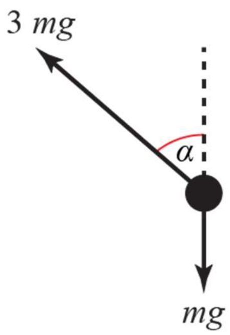
Using equations **(1)** and **(2)** gives:

$\dfrac{mv^2}{40(\sin 25° + 0.6\cos 25°)} = \dfrac{mg}{\cos 25° - 0.6\sin 25°}$

$\Rightarrow v^2 = \dfrac{40g(\sin 25° + 0.6\cos 25°)}{\cos 25° - 0.6\sin 25°} = 580.37$ (2 d.p.)

$\Rightarrow v = 24\,\text{m}\,\text{s}^{-1}$ (2 s.f.)

5 a The forces acting on the metal ball are the weight of the ball and tension along the string.



R(↑): $3mg\cos\alpha - mg = 0$

$\Rightarrow \cos\alpha = \frac{mg}{3mg} = \frac{1}{3}$

$\Rightarrow \alpha = 70.5°$ (3 s.f.)

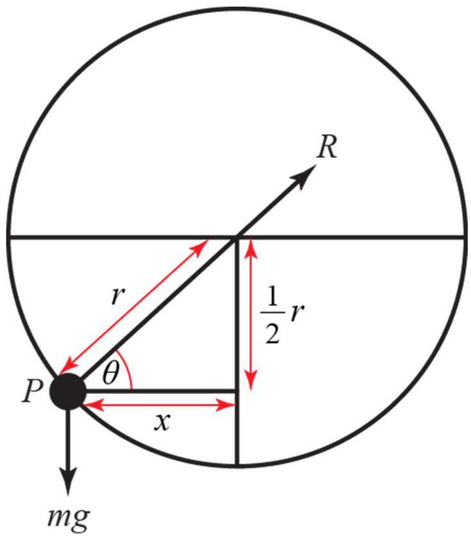
b R(←): $3mg\sin\alpha = mr2gk$ using $F = ma$ and $a = r\omega^2$

Let the length of the string be l, then from ΔAOB is is clear that $\sin\alpha = \frac{r}{l}$

So $3mg\frac{r}{l} = mr2gk$

$\Rightarrow l = \frac{3}{2k}$

6 a The forces acting on the particle are its weight and the normal reaction.



Let θ be the angle between the normal reaction and the horizontal.

Then $\sin\theta = \frac{\frac{1}{2}r}{r} = \frac{1}{2} \Rightarrow \theta = 30°$

R(↑): $R\sin 30° - mg = 0$

$\Rightarrow R = \frac{mg}{\sin 30°} = 2mg$

6 b $R(\rightarrow): R\cos 30° = mx\omega^2$ using $F = ma$ and $a = r\omega^2$

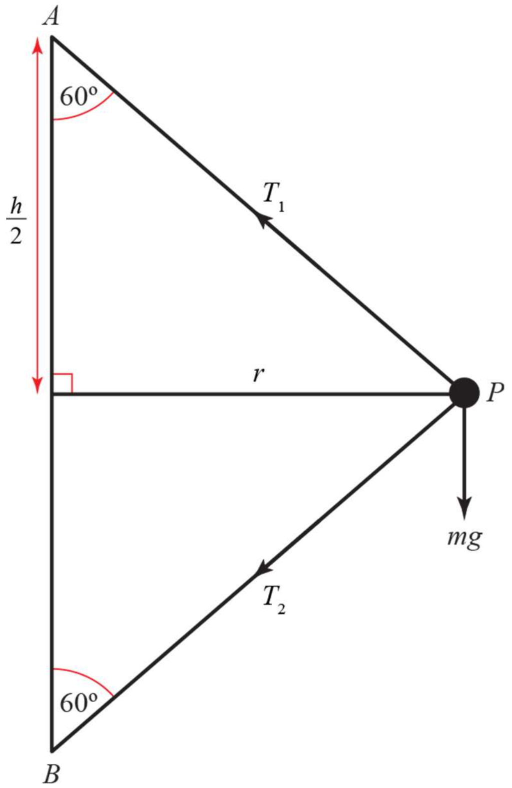
Using the result from part **a** and as $x = r\cos 30°$ this gives:

$2mg = mr\omega^2$

$\Rightarrow \omega = \sqrt{\frac{2g}{r}}$

Time to complete one revolution $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{2g}}$

7 a Let T_1 be the tension in AP and T_2 be the tension in BP. The forces acting on P are the tensions in the two strings and its weight.



From the diagram, it can be seen that the equilateral triangle APB can be divided into two right-angled triangles, where:

$\tan 60° = \frac{r}{\frac{h}{2}}$

$\Rightarrow r = \frac{h}{2} \times \tan 60° = \frac{\sqrt{3}h}{2}$

7 b Resolving the forces from the diagram in part **a**

$R(\uparrow): T_1 \cos 60° - T_2 \cos 60° - mg = 0$

$\Rightarrow T_1 - T_2 = 2mg$ **(1)**

$R(\leftarrow): T_1 \sin 60° + T_2 \sin 60° = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow \frac{\sqrt{3}}{2}T_1 + \frac{\sqrt{3}}{2}T_2 = m\frac{\sqrt{3}}{2}h\omega^2$ as $r = \frac{\sqrt{3}}{2}h$ from part **a**

$\Rightarrow T_1 + T_2 = mh\omega^2$ **(2)**

Adding equations **(1)** and **(2)** gives: $2T_1 = 2mg + mh\omega^2 \Rightarrow T_1 = mg + \frac{1}{2}mh\omega^2$

Substituting for T_1 in equation **(2)** gives: $T_2 = \frac{1}{2}mh\omega^2 - mg$

c Both strings are taut, therefore $T_1 > 0$ and $T_2 > 0$. From part **b**, $T_1 > 0$ for all values of ω

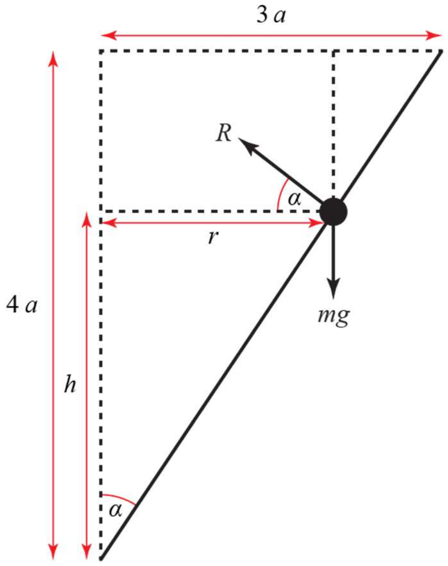
From the formula for T_2 in part **b**, the condition that $T_2 > 0 \Rightarrow \omega > \sqrt{\frac{2g}{h}}$

As $T = \frac{2\pi}{\omega}$, $\omega = \frac{2\pi}{T}$

So $\omega > \sqrt{\frac{2g}{h}} \Rightarrow \frac{2\pi}{T} > \sqrt{\frac{2g}{h}} \Rightarrow T < 2\pi\sqrt{\frac{h}{2g}}$ as $T > 0$ and $\sqrt{\frac{h}{2g}} < 0$

$\Rightarrow T < \pi\sqrt{\frac{2h}{g}}$ as required

8 Let α be the angle between the slant side and the axis of the cone, r the radius of the horizontal circle with centre C and h the height of C above V. The forces acting on the particle are its weight and the normal reaction.



From the diagram, $\tan\alpha = \frac{3a}{4a} = \frac{3}{4}$ and $\tan\alpha = \frac{r}{h}$

R($\uparrow$): $R\sin\alpha - mg = 0$

$\Rightarrow R\sin\alpha = mg$ **(1)**

R($\leftarrow$): $R\cos\alpha = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow R\cos\alpha = \frac{mr8g}{9a}$ **(2)** using $\omega = \sqrt{\frac{8g}{9a}}$

Dividing equation **(1)** by equation **(2)** gives:

$\tan\alpha = mg \div \frac{8mrg}{9a} = \frac{9a}{8r}$

$\Rightarrow \frac{3}{4} = \frac{9a}{8r}$ using $\tan\alpha = \frac{3}{4}$

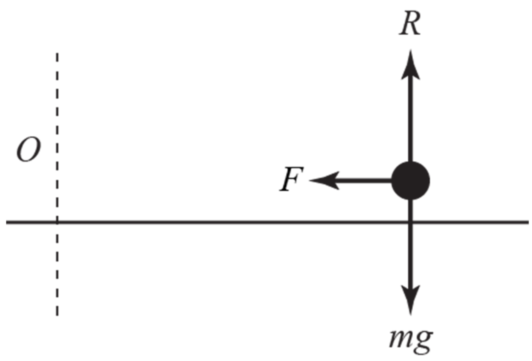
$\Rightarrow r = \frac{3a}{2}$

As $\tan\alpha = \frac{3}{4} = \frac{r}{h}$, $h = \frac{4}{3}r$

So $h = \frac{4}{3} \times \frac{3a}{2} = 2a$

The height of C above V is $2a$.

9 a The forces acting on the particle are its weight, the normal reaction and friction.



R(↑): $R - mg = 0 \Rightarrow R = mg$

R(←): $F = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow F = \dfrac{4ma\omega^2}{3}$ as $r = \dfrac{4}{3}a$

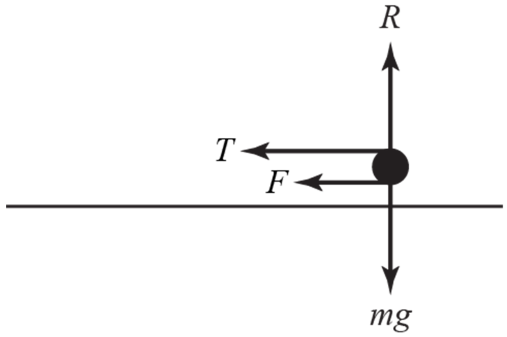
As P remains at rest $F \leqslant \mu R$

So $\dfrac{4ma\omega^2}{3} \leqslant \dfrac{3}{5}R$

$\Rightarrow \dfrac{4ma\omega^2}{3} \leqslant \dfrac{3mg}{5}$

$\Rightarrow \omega^2 \leqslant \dfrac{9g}{20a}$ as required

9 b The forces now acting on the particle are its weight, the normal reaction, friction and the tension in the elastic string.



Find T using Hooke's law (Further Mechanics 1, Chapter 3): $T = \dfrac{\lambda x}{l}$,

where λ is the modulus of elasticity, x is the extension of the string and l is its natural length

So in this case, $T = \dfrac{2mg}{a} \times \dfrac{a}{3} = \dfrac{2mg}{3}$

R($\leftarrow$): $T + F = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow \omega^2 = \dfrac{3}{4ma}\left(\dfrac{2mg}{3} + F\right)$

Note that the frictional force can act away from O against the pull of the elastic string, or towards O through the force of the acceleration of the particle generated by the circular motion. As P remains at rest $-\mu R \leqslant F \leqslant \mu R$ (where $\mu R = 0.6mg$ from part **a**).

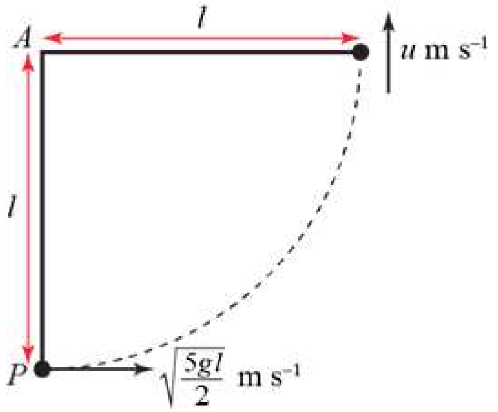
So, from the equation for ω^2, it is maximum when $F = \mu R = \dfrac{3mg}{5}$

So $\omega^2_{\text{max}} = \dfrac{3}{4ma}\left(\dfrac{2mg}{3} + \dfrac{3mg}{5}\right) = \dfrac{3}{4ma} \times \dfrac{19mg}{15} = \dfrac{19g}{20a}$

Similarly ω^2 is minimum when $F = -\mu R = -\dfrac{3mg}{5}$

So $\omega^2_{\text{min}} = \dfrac{3}{4ma}\left(\dfrac{2mg}{3} - \dfrac{3mg}{5}\right) = \dfrac{3}{4ma} \times \dfrac{mg}{15} = \dfrac{g}{20a}$

10 a Let u be the speed of P when the string is horizontal.



Applying the work-energy principle, the sum of the particle's kinetic and gravitational potential energy remains constant, so when it reaches the horizontal the loss of kinetic energy = the gain in potential energy.

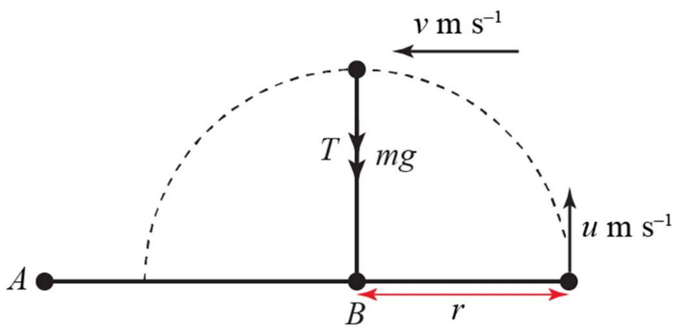
Take the starting point as the zero level for potential energy.

So $\frac{1}{2}m\frac{5gl}{2} - \frac{1}{2}mu^2 = mgl$

$\Rightarrow u^2 = \frac{5gl}{2} - 2gl = \frac{gl}{2}$

$\Rightarrow u = \sqrt{\frac{gl}{2}}$

10 b Let the particle move in a semi-circle about B with radius r.



Taking the line AB as the zero level for potential energy, then applying conservation of energy at the highest point of the semi-circle:

$$\frac{1}{2}m(u^2 - v^2) = mgr$$

$$\Rightarrow v^2 = u^2 - 2gr \qquad (1)$$

Resolving the vertical forces at the highest point:

$$\text{R}(\downarrow): T + mg = \frac{mv^2}{r} \qquad \text{using } F = ma \text{ and } a = \frac{v^2}{r}$$

$$\Rightarrow T = \frac{m(u^2 - 2gr)}{r} - mg \qquad \text{substituting for } v^2 \text{ using equation (1)}$$

$$\Rightarrow T = \frac{mu^2}{r} - 3mg$$

$$\Rightarrow T = \frac{mgl}{2r} - 3mg \qquad \text{substituting for } u^2 \text{ using result from part } \mathbf{a}$$

As the string does not go slack $T > 0$, so

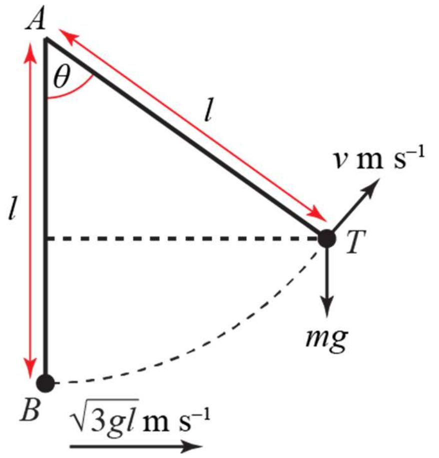
$$\frac{mgl}{2r} - 3mg > 0$$

$$\Rightarrow mgl > 6mgr$$

$$\Rightarrow r < \frac{l}{6}$$

As $AB = l - r$, this shows that $AB > \frac{5l}{6}$

11 a Let the speed of the particle be v when it makes an angle θ with the downward vertical.



Take the horizontal through B as the zero level for potential energy. Using conservation of energy:

$\frac{1}{2}m(3gl - v^2) = mgl(1 - \cos\theta)$

$\Rightarrow v^2 = 3gl - 2gl(1 - \cos\theta)$

$\Rightarrow v^2 = gl + 2gl\cos\theta$ **(1)**

Resolving along the string:

R (↖): $T - mg\cos\theta = \frac{mv^2}{l}$ using $F = ma$ and $a = \frac{v^2}{r}$ and $r = l$

$\Rightarrow T = mg\cos\theta + \frac{mgl + 2gl\cos\theta}{l}$ substituting for v^2 using equation **(1)**

$\Rightarrow T = mg + 3mg\cos\theta = mg(1 + 3\cos\theta)$ as required

b The instant the string becomes slack, $T = 0$.

Using the expression for T from part **a**, this occurs when:

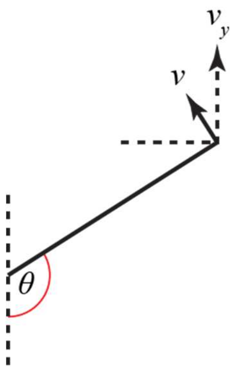
$1 + 3\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{3}$

Substituting for $\cos\theta$ in equation **(1)** gives:

$v^2 = gl + 2gl \times -\frac{1}{3} = \frac{gl}{3}$

$\Rightarrow v = \sqrt{\frac{gl}{3}}$

11 c The particle now moves as a projectile under gravity. The maximum height is achieved when the vertical component of the velocity is zero.



$$v_y = v\sin(180° - \theta) = \sqrt{\frac{gl}{3}}\sin\theta$$

If $\frac{\pi}{2} < \theta < \pi$ and $\cos\theta = -\frac{1}{3}$, then $\sin\theta = \frac{2\sqrt{2}}{3}$

So $v_y = \frac{2\sqrt{2}}{3}\sqrt{\frac{gl}{3}}$

Considering the particle's vertical motion, $u = v_y$, $v = 0$, $a = -g$, $s = h$, where h is the height above the point the string becomes slack.

Using $v^2 = u^2 - 2gh$

$$h = \frac{v_y^2}{2g} = \frac{8}{9} \times \frac{gl}{3} \times \frac{1}{2g} = \frac{4l}{27}$$

The height at which string becomes slack $= l + l\cos(180° - \theta) = l(1 - \cos\theta) = \frac{4l}{3}$

So if H is the maximum height above the level of B reached by P

$$H = \frac{4l}{3} + \frac{4l}{27} = \frac{40l}{27}$$

12 a Take the horizontal through l as the zero level for potential energy. When angle $\theta = \alpha$, the particle is momentarily at rest. Using conservation of energy, with the loss of kinetic energy equal to the gain in potential energy:

$$\frac{1}{2}mu^2 = mgl(1 - \cos\alpha) = \frac{mgl}{3} \qquad \text{as } \cos\alpha = \frac{2}{3}$$

$$\Rightarrow u^2 = \frac{2}{3}gl$$

$$\Rightarrow u = \sqrt{\frac{2gl}{3}}$$

12 b Let the speed of the particle at θ be v.
Resolving along the string gives:

R (↖): $T - mg\cos\theta = \frac{mv^2}{l}$ using $F = ma$ and $a = \frac{v^2}{r}$ and $r = l$

$\Rightarrow T = mg\cos\theta + \frac{mv^2}{l}$

Applying conservation of energy:

$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgl(1-\cos\theta)$

$\Rightarrow v^2 = u^2 - 2gl(1-\cos\theta) = \frac{2gl}{3} - 2gl(1-\cos\theta)$ using the result from part **a**

$\Rightarrow v^2 = 2gl\cos\theta - \frac{4gl}{3}$

Substituting for v^2 in the equation for T gives:

$T = mg\cos\theta + 2mg\cos\theta - \frac{4mg}{3}$

$= 3mg\cos\theta - \frac{4mg}{3}$

$= \frac{mg}{3}(9\cos\theta - 4)$

c Maximum value of T is when $\cos\theta = 1$

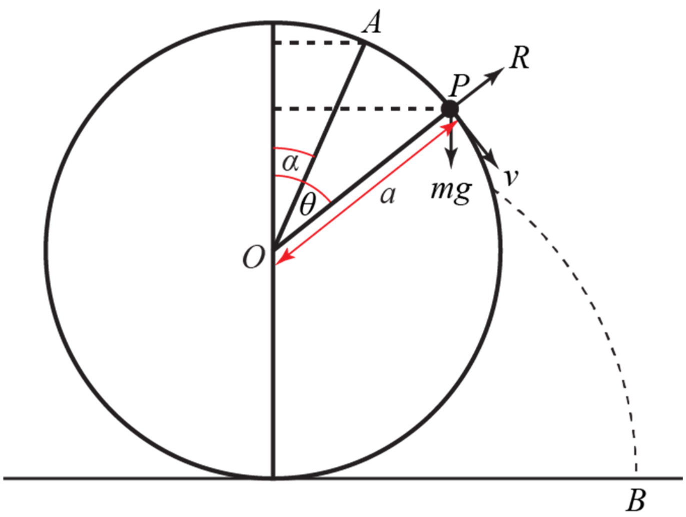
So $T_{\text{max}} = \frac{5mg}{3}$

Minimum value of T is when $\cos\theta = \frac{2}{3}$

So $T_{\text{min}} = \frac{2mg}{3}$

Hence $\frac{2mg}{3} \leqslant T \leqslant \frac{5mg}{3}$

13 a This is a diagram of the problem.



Take the horizontal through A as the zero level for potential energy. Using conservation of energy, the gain of kinetic energy at P equals the loss in potential energy:

$$\frac{1}{2}mv^2 = mg(a\cos\alpha - a\cos\theta)$$
$$v^2 = 2ga(\cos\alpha - \cos\theta)$$

b Resolving along the radius OP:

R (↙): $mg\cos\theta - R = \dfrac{mv^2}{a}$

At the point when P loses contact with the sphere $R = 0$

$\Rightarrow mg\cos\theta = \dfrac{mv^2}{a} = 2gm(\cos\alpha - \cos\theta)$ substituting for v^2 from part **a**

$\Rightarrow 3\cos\theta = 2\cos\alpha = \dfrac{3}{2}$

$\Rightarrow \cos\theta = \dfrac{1}{2}$, so $\theta = 60°$ or $\dfrac{\pi}{3}$ radians

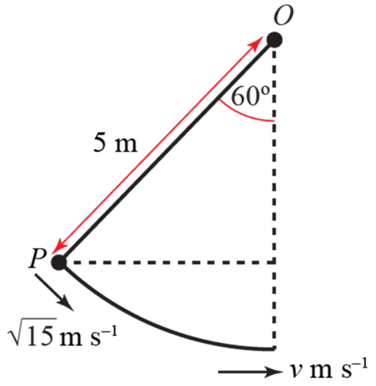
c Let the speed of P when it hits the table be w. Then taking the horizontal through A as the zero level for potential energy and using conservation of energy, the gain of kinetic energy at B equals the loss in potential energy:

$$\frac{1}{2}mw^2 = mg(a + a\cos\alpha)$$
$$\Rightarrow w^2 = 2ga\left(1 + \frac{3}{4}\right) = \frac{7ga}{2}$$

So $w = \sqrt{\dfrac{7ga}{2}}\ \text{m s}^{-1}$

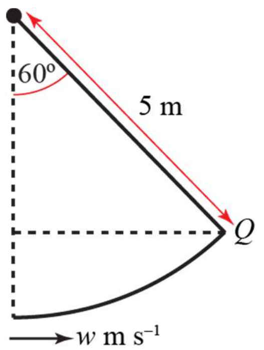
There are alternative ways to calculate w that use projectiles, but the approach shown above is the shortest method.

14 a Let the speed of the trapeze artist at the lowest point of her path be v. Then taking the horizontal through A as the zero level for potential energy and using conservation of energy, the gain in kinetic energy at this point equals the loss in potential energy:



$$\frac{1}{2}mv^2 - \frac{1}{2}m(\sqrt{15})^2 = mg \times 5(1-\cos 60°)$$
$$\Rightarrow v^2 = 15 + 5g = 64$$
$$\Rightarrow v = 8\,\text{m s}^{-1}$$

b Let the velocity after catching the ball be w. Then the loss in kinetic energy at point Q where the trapeze artist becomes momentarily stationery equals the gain in potential energy from the lowest point of the trapeze artist's path.



$$\frac{1}{2}(60+m)w^2 - 0 = (60+m)g5(1-\cos 60°)$$
$$\Rightarrow w^2 = 5g = 49, \text{ so } w = 7\,\text{m s}^{-1}$$

So at the instant just prior to catching the ball, the trapeze artist is travelling at $8\,\text{m s}^{-1}$ (part **a**) and the ball is travelling at $3\,\text{m s}^{-1}$ in the opposite direction, and immmediately after catching the ball she is travelling at $7\,\text{m s}^{-1}$. Using conservation of linear momentum at the instant when she catches the ball, this gives:

$$60 \times 8 + (-3 \times m) = (60+m) \times 7$$
$$\Rightarrow 480 - 3m = 420 + 7m$$
$$\Rightarrow 10m = 60$$
So $m = 6$ kg

c $\text{R}(\uparrow): T - 66g = \frac{66w^2}{r}$ using $F = ma$ and $a = \frac{v^2}{r}$

$$\Rightarrow T = 66g + 66 \times \frac{7^2}{5} = 1293.6 = 1300\text{ N (2 s.f.)}$$

15 a Let the speed at C be $v\,\text{m s}^{-1}$. Then taking the horizontal through B as the zero level for potential energy and using conservation of energy, the gain in kinetic energy at point C equals the loss in potential energy:

$$\frac{1}{2}mv^2 - \frac{1}{2}m\times 20^2 = mg\times 50(1-\cos 60°)$$
$$\Rightarrow v^2 = 20^2 + 50g \Rightarrow v^2 = 890$$

So $v = 30\,\text{m s}^{-1}$ (2 s.f.)

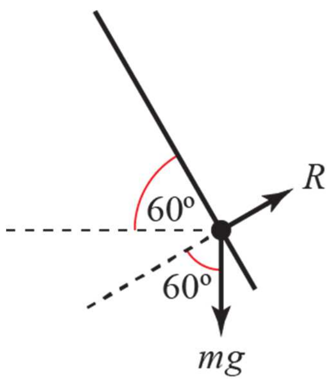
b At C, R($\uparrow$): $R - 70g = \frac{70v^2}{50}$ using $F = ma$ and $a = \frac{v^2}{r}$

$$\Rightarrow R = 70g + \frac{70\times 890}{50} = 686 + 1246 = 1900\,\text{N (2 s.f.)}$$

c Consider motion C to D. Let the speed at D be $w\,\text{m s}^{-1}$. Then taking the horizontal through C as the zero level for potential energy and using conservation of energy, the loss in kinetic energy at point C equals the gain in potential energy:

$$\frac{1}{2}m\times 890 - \frac{1}{2}mw^2 = mg\times 50(1-\cos 30°)$$
$$\Rightarrow w^2 = 890 - 100g(1-\cos 30°) = 759 \text{ (3 s.f.)}$$
$$\Rightarrow w = 28\,\text{m s}^{-1} \text{ (2 s.f.)}$$

d Resolving perpendicular to the slope at B just before the circular motion and just after circular motion begins:



Before: $R = mg\cos 60° = 35g$

After: $R - mg\cos 60° = \frac{m\times 20^2}{50} \Rightarrow R = 35g + 560$

So change in $R = 560\,\text{N}$

e Allowing for the influence of friction would mean that the skier would arrive at C with lower speed. From the equation used in part **b**, this would result in a lower normal reaction.

16 a Let the velocity of the particle at C be v. Taking the horizontal through A as the zero level for potential energy and using conservation of energy, the gain in kinetic energy at C equals the loss in potential energy:

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga(1-\cos\theta)$$

$$\Rightarrow v^2 = u^2 + 2ga(1-\cos\theta) \qquad \textbf{(1)}$$

Resolving along the radius through C:

R (↙): $-R + mg\cos\theta = \frac{mv^2}{a}$

As the particle leaves the sphere at C, $R = 0$ so $v^2 = ag\cos\theta$

Substituting for v^2 in equation **(1)** gives:

$$ag\cos\theta = u^2 + 2ga(1-\cos\theta)$$

$$\Rightarrow 3\cos\theta = \frac{u^2}{ag} + 2$$

$$\Rightarrow \cos\theta = \frac{2}{3} + \frac{u^2}{3ag}$$

b Using conservation of energy, the gain in kinetic energy from C to when the particle hits the ground equals the loss in potential energy:

$$\frac{1}{2}m\left(\frac{9ag}{2}\right) - \frac{1}{2}m(ag\cos\theta) = mga(1+\cos\theta)$$

$$\Rightarrow \frac{3}{2}\cos\theta = \frac{9}{4} - 1 = \frac{5}{4} \Rightarrow \cos\theta = \frac{5}{6}$$

So $\theta = 34°$ (2 s.f.)

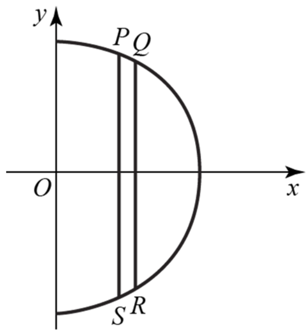
17 The centre of mass lies on the axis of symmetry, $y = 0$.

To find the x coordinate, using $\bar{x} = \frac{\int y^2 x\,dx}{\int y^2\,dx}$, and substituting $y = \sqrt{x}$ gives:

$$\bar{x} = \frac{\int_0^4 x^2\,dx}{\int_0^4 x\,dx} = \frac{\left[\frac{1}{3}x^3\right]_0^4}{\left[\frac{1}{2}x^2\right]_0^4} = \frac{64}{3} \div 8 = \frac{8}{3}$$

So the coordinates of the centre of mass of the solid are $\left(\frac{8}{3}, 0\right)$, a distance of $\frac{8}{3}$ from O.

18 a Let the straight edge lie along the y-axis. Then the centre of mass lies on the x-axis from symmetry.



If P has coordinates (x, y) and the elemental strip $PQRS$ has width δx then its area is $2y\delta x$.

The mass M of the lamina $= \frac{1}{2}\pi a^2\rho$, where ρ is the mass per unit area of the lamina.

Let $\bar{x}$ be the distance of the centre of mass from O, then $M\bar{x} = \int_0^a 2\rho xy\,\mathrm{d}x$

As the boundary of the semicircle has the equation $x^2 + y^2 = a^2$, then $y = (a^2 - x^2)^{\frac{1}{2}}$

So $M\bar{x} = \int_0^a 2\rho x(a^2 - x^2)^{\frac{1}{2}}\,\mathrm{d}x = \left[-\frac{2}{3}\rho(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}\rho a^3$

$\Rightarrow \bar{x} = \dfrac{\frac{2}{3}\rho a^3}{\frac{1}{2}\pi a^2\rho} = \dfrac{4a}{3\pi}$ as required

b Using the result from part **a** and letting $\bar{x}$ be the distance of the centre of mass of the resulting lamina from O:

Shape	Mass	Distance of centre of mass from O
Semicircle radius a	$\frac{1}{2}\pi\rho a^2$	$\frac{4a}{3\pi}$
Semicircle radius b	$\frac{1}{2}\pi\rho b^2$	$\frac{4b}{3\pi}$
Resulting	$\frac{1}{2}\pi\rho(a^2 - b^2)$	$\bar{x}$

Taking moments about O:

$\frac{1}{2}\pi\rho(a^2 - b^2)\bar{x} = \frac{1}{2}\pi\rho a^2 \times \frac{4a}{3\pi} - \frac{1}{2}\pi\rho b^2 \times \frac{4b}{3\pi}$

$\Rightarrow \bar{x} = \dfrac{4}{3\pi}\dfrac{(a^3 - b^3)}{(a^2 - b^2)} = \dfrac{4}{3\pi}\dfrac{(a-b)(a^2 + ab + b^2)}{(a-b)(a+b)} = \dfrac{4}{3\pi}\dfrac{(a^2 + ab + b^2)}{(a+b)}$ as required

c As $b \to a$, the area becomes a circular arc and from the equation found in part **b**

$\bar{x} \to \dfrac{4}{3\pi} \times \dfrac{(a^2 + a^2 + a^2)}{(a+a)} = \dfrac{4}{3\pi} \times \dfrac{3a^2}{2a} = \dfrac{2a}{\pi}$

19 a Use $\bar{x} = \dfrac{\int y^2 x\,\mathrm{d}x}{\int y^2\,\mathrm{d}x}$, and substitute $y = \dfrac{1}{2x^2}$

$$\bar{x} = \frac{\int_1^2 xy^2\mathrm{d}x}{\int_1^2 y^2\mathrm{d}x} = \frac{\int_1^2 x\times\frac{1}{4}x^{-4}\mathrm{d}x}{\int_1^2 \frac{1}{4}x^{-4}\mathrm{d}x} = \frac{\int_1^2 x^{-3}\mathrm{d}x}{\int_1^2 x^{-4}\mathrm{d}x}$$

$$= \frac{\left[-\frac{1}{2}x^{-2}\right]_1^2}{\left[-\frac{1}{3}x^{-3}\right]_1^2} = \frac{3\left(1^{-2}-2^{-2}\right)}{2\left(1^{-3}-2^{-3}\right)} = \frac{3}{2}\times\frac{3}{4}\times\frac{8}{7} = \frac{9}{7}$$

The centre of mass is $\frac{9}{7}$ m from the y-axis, hence $\left(\frac{9}{7}-1\right)=\frac{2}{7}$ m from the larger plane face.

b Let ρ be the mass per unit volume of the hemisphere H and the solid S.

Then the mass of the $S = \int_1^2 \pi y^2\rho\mathrm{d}x = \int_1^2 \frac{\pi}{4}x^{-4}\rho\mathrm{d}x = \left[-\frac{\pi\rho}{12}x^{-3}\right]_1^2 = \frac{\pi\rho}{12}\left(1-\frac{1}{8}\right) = \frac{7\pi\rho}{96}$

The mass of the hemisphere and its distance of its centre of mass from its plane face can be found from the standard formulae, using $r = 0.5$.

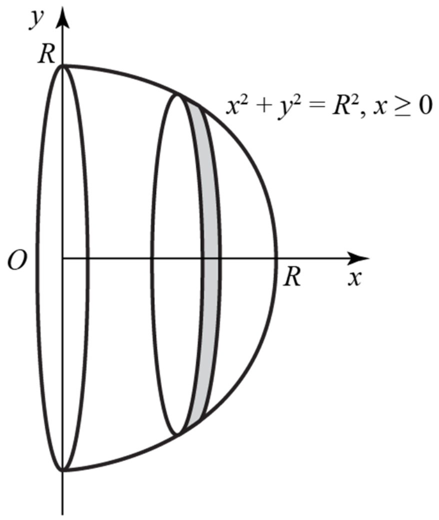
Shape	Mass	Distance of centre of mass from trophy's plane face
Solid, S	$\frac{7\pi\rho}{96}$	$1-\frac{2}{7}=\frac{5}{7}$
Hemisphere, H	$\frac{2\pi\rho}{3}\left(\frac{1}{2}\right)^3=\frac{\pi\rho}{12}$	$1+\frac{3}{8}\times\frac{1}{2}=\frac{19}{16}$
Trophy, T	$\pi\rho\left(\frac{7}{96}+\frac{1}{12}\right)=\frac{5\pi\rho}{32}$	$\bar{x}$

Taking moments around the trophy's plane face (the bottom of the trophy as shown):

$$\frac{5\pi\rho}{32}\bar{x} = \frac{\pi\rho}{12}\times\frac{19}{16}+\frac{7\pi\rho}{96}\times\frac{5}{7}$$

$$\bar{x} = \frac{32}{5}\left(\frac{19}{192}+\frac{5}{96}\right) = \frac{32}{5}\times\frac{29}{192} = \frac{29}{30} = 0.967\text{ m (3 s.f.)}$$

20 a A hemisphere is generated when a semicircle is notated through 180° about the x-axis.



Divide the hemisphere into circular discs, with each disc having mass $\rho\pi y^2\delta x$ and centre of mass at a distance x from O.

So $\bar{x} = \dfrac{\int_0^R \rho\pi x y^2 \, dx}{\int_0^R \rho\pi y^2 \, dx} = \dfrac{\int_0^R x(R^2 - x^2)\, dx}{\int_0^R (R^2 - x^2)\, dx}$

$$= \frac{\left[\frac{1}{2}R^2x^2 - \frac{1}{4}x^4\right]_0^R}{\left[R^2x - \frac{1}{3}x^3\right]_0^R} = \frac{\frac{1}{2}R^4 - \frac{1}{4}R^4}{R^3 - \frac{1}{3}R^3} = \frac{3}{2} \times \frac{1}{4}R = \frac{3}{8}R$$

b Using the standard formulae for a cone:

Shape	Mass	Distance of centre of mass from V
Cone	$\frac{1}{3}\pi a^2 ka\rho$	$\frac{3}{4}ka$
Hemisphere	$\frac{2}{3}\pi a^3\rho$	$ka + \frac{3}{8}a$
Top	$\frac{1}{3}\pi a^3\rho(k+2)$	$\bar{x}$

Taking moments about V:

$$\frac{1}{3}\pi a^3\rho(k+2)\bar{x} = \frac{1}{3}\pi a^3\rho k\left(\frac{3}{4}ka\right) + \frac{2}{3}\pi a^3\rho\left(ka + \frac{3a}{8}\right)$$

$$(k+2)\bar{x} = \frac{3}{4}k^2a + 2ka + \frac{3a}{4}$$

$$\bar{x} = \frac{(3k^2 + 8k + 3)a}{4(k+2)}$$

20 c The manufacturer's requirement is that $\bar{x} = ka$

Hence from part **b** $\frac{3k^2 + 8k + 3}{4(k+2)} = k$

$\Rightarrow 3k^2 + 8k + 3 = 4k^2 + 8k$

$\Rightarrow k^2 = 3$, so $k = \sqrt{3}$

21 a Using the standard results for a hemisphere:

Shape	Mass	Ratio of masses	Distance of centre of mass from *O*
Large hemisphere	$\frac{2}{3}\pi a^3 \rho$	8	$\frac{3}{8}a$
Small hemisphere	$\frac{2}{3}\pi\left(\frac{a}{2}\right)^3 \rho$	1	$\frac{3}{16}a$
Remainder	$\frac{2}{3}\pi\frac{7a^3}{8}\rho$	7	$\bar{x}$

Taking moments about O:

$$7\bar{x} = 8 \times \frac{3}{8}a - 1 \times \frac{3}{16}a$$

$$\bar{x} = \frac{1}{7} \times \frac{45}{16}a = \frac{45a}{112}$$

b Using the result from part **a**:

	Mass ratios	Distance of centre of mass from *O*
Bowl	M	$\frac{45}{112}a$
Liquid	kM	$\frac{3}{16}a$
Bowl + liquid	$(k+1)M$	$\frac{17}{48}a$

Taking moments about O:

$$(k+1)M \times \frac{17}{48}a = M \times \frac{45}{112}a + kM \times \frac{3}{16}a$$

$$k\left(\frac{17}{48} - \frac{3}{16}\right) = \frac{45}{112} - \frac{17}{48}$$

$$\frac{8}{48}k = \frac{45}{112} - \frac{17}{48}$$

$$k = 6 \times \left(\frac{45}{112} - \frac{17}{48}\right) = \frac{6(2160 - 1906)}{5376} = \frac{254}{896} = \frac{2}{7}$$

22 a $V = \int_0^1 \pi y^2 \mathrm{d}x = \int_0^1 \frac{\pi}{4}(x-2)^4 \mathrm{d}x$ substituting $y = \frac{1}{2}(x-2)^2$

$\Rightarrow V = \left[\frac{\pi}{20}(x-2)^5\right]_0^2 = -\frac{\pi}{20}(-2)^5 = \frac{32\pi}{20} = \frac{8\pi}{5}\text{cm}^3$

b Let $\bar{x}$ be the distance of the centre of mass of S from its plane face

To find, using $M\bar{x} = \int \rho\pi y^2 x \,\mathrm{d}x$, and substituting $y^2 = \frac{1}{4}(x-2)^2$ and $M = \frac{8\pi\rho}{5}$ from part **a** gives:

$\bar{x} = \frac{5}{32}\int_0^2 (x-2)^4 x \mathrm{d}x$

Integrate using the substitution $u = x - 2$

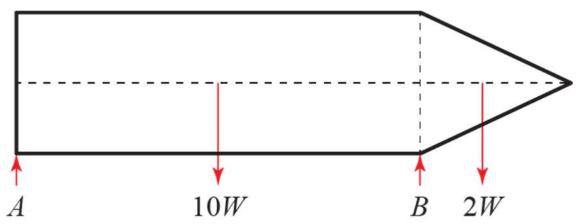
$\bar{x} = \frac{5}{32}\int_{-2}^0 u^4(u+2)\mathrm{d}u = \frac{5}{32}\int_{-2}^0 u^5 + 2u^4 \mathrm{d}u$

$= \frac{5}{32}\left[\frac{1}{6}u^6 + \frac{2}{5}u^5\right]_{-2}^0 = \frac{5}{32}\left(\frac{2}{5}\times 2^5 - \frac{1}{6}2^6\right) = \frac{5}{32}\left(\frac{64}{5} - \frac{64}{6}\right)$

$= 10\left(\frac{1}{5} - \frac{1}{6}\right) = \frac{10}{30} = \frac{1}{3}$

The centre of mass lies on the axis of symmetry at a distance of $\frac{1}{3}$cm from the plane base.

c Let the reaction force at A be A, and at B be B.



Taking moments about B:

$A \times 8 + 2W \times \frac{1}{3} = 10W \times 4$

$A = \frac{\left(40 - \frac{2}{3}\right)W}{8} = \frac{118W}{24} = \frac{59W}{12}$

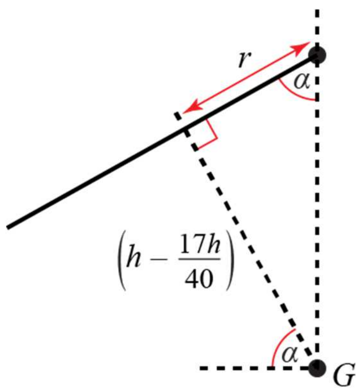
23 a Using the formulae for standard uniform bodies:

Shape	Mass	Mass ratios	Distance of centre of mass from base
Cylinder	$\pi r^2 h\rho$	1	$\frac{h}{2}$
Cone	$\frac{1}{3}\pi r^2 \frac{h}{2}\rho$	$\frac{1}{6}$	$h-\frac{1}{4}\left(\frac{h}{2}\right)=\frac{7h}{8}$
Ornament	$\frac{5}{6}\pi r^2 h\rho$	$\frac{5}{6}$	$\bar{x}$

Take moments about O, the centre of plane base:

$$\frac{5}{6}\bar{x}=1\times\frac{h}{2}-\frac{1}{6}\times\frac{7h}{8}$$
$$\frac{5}{6}\bar{x}=\frac{h}{2}-\frac{7h}{48}=\frac{17h}{48}$$
$$\bar{x}=\frac{17h}{48}\times\frac{6}{5}=\frac{17h}{40}$$

b The centre of mass G of the ornament will be directly below the point of suspension.



$$\tan\alpha=\frac{h-\frac{17h}{40}}{r}=\frac{23h}{40r}$$

As $h=4r$, this gives $\tan\alpha=\frac{23r}{10r}=2.3$

$\Rightarrow \alpha=66.5°$ (1 d.p.)

24 a Let ρ be the mass per unit area of the material and $\bar{x}$ the distance of the centre of mass of the closed container from O. Using the formulae for standard uniform bodies:

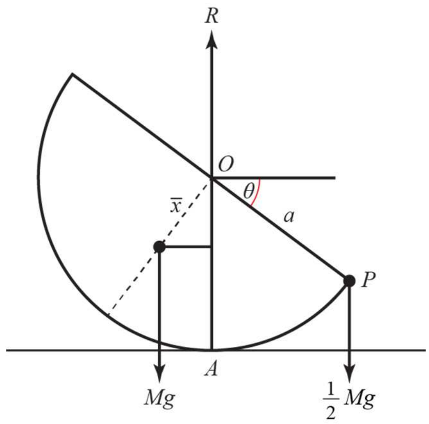
Shape	Mass	Distance of centre of mass from *O*
Circular disc	$\pi a^2 \rho$	0
Hemispherical bowl	$2\pi a^2 \rho$	$\frac{1}{2}a$
Closed container	$3\pi a^2 \rho$	$\bar{x}$

Taking moments about O:

$$3\pi a^2 \rho \bar{x} = 0 + 2\pi a^2 \rho \times \frac{a}{2}$$

$$\bar{x} = \frac{a}{3}$$

b The container is resting in equilibrium, so the weight of P acts to one side and the weight of C balances on the other side.



Taking moments about O:

$$Mg \times \frac{a}{3}\sin\theta = \frac{1}{2}Mg \times a\cos\theta$$

So $\tan\theta = \frac{3}{2}$

$\Rightarrow \theta = 56°$ (to the nearest degree)

25 a The hemisphere K has mass M. The hemisphere H has double the radius, so it has 8 times the volume. Its mass is $8M$. Therefore the composite body S has mass $M+8M=9M$.

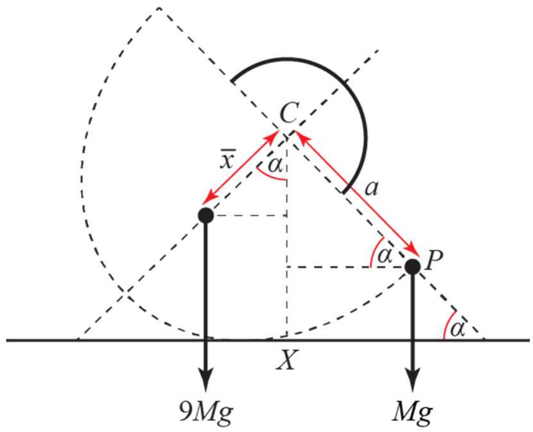
Shape	Mass	Distance of centre of mass from C
H	$8M$	$\frac{3a}{8}$
K	M	$\frac{-3a}{16}$
S	$M+8M$	$\bar{x}$

Taking moments about C:

$$9M\bar{x}=8M\times\frac{3a}{8}-M\times\frac{3a}{16}=M\times\frac{45a}{16}$$

$$\bar{x}=\frac{45a}{9\times16}=\frac{5a}{16}$$

b The composite body with particle P attached rests in equilibrium, with C above the point of contact with the plane, X.



$Mg\times a\cos\alpha=9Mg\bar{x}\sin\alpha=9\times\frac{5}{16}Mg\sin\alpha$ substituting result for $\bar{x}$ from part **a**

$$\Rightarrow\tan\alpha=\frac{16}{45}$$

26 a Let ρ be the mass per unit volume of the material of the cylinder and $\bar{x}$ the distance of the centre of mass of the toy from O. Using the formulae for standard uniform bodies:

Shape	Mass	Mass ratio	Distance of centre of mass from O
Hemisphere	$\frac{2}{3}\pi r^3 6\rho$	$4r$	$\frac{5r}{8}$
Cylinder	$\pi r^2 h\rho$	h	$\frac{h}{2}+r$
Toy	$\pi r^2\rho(4r+h)$	$4r+h$	$\bar{x}$

Taking moments about O:

$$(4r+h)\bar{x} = 4r\times\frac{5r}{8}+h\left(\frac{h}{2}+r\right)$$

$$\bar{x}=\frac{5r^2+h^2+2rh}{2(4r+h)}=\frac{h^2+2hr+5r^2}{2(h+4r)} \qquad \text{as required}$$

b If the toy remains in equilibrium when resting on any point of the curved surface, its centre of mass must be in the centre of the hemisphere's flat surface, so $\bar{x}=r$.

Hence from part **a** $\frac{h^2+2hr+5r^2}{2(h+4r)}=r$

$$\Rightarrow h^2+2hr+5r^2=2rh+8r^2$$

$$\Rightarrow h^2=3r^2$$

$$\Rightarrow h=\sqrt{3}r$$

27 a Let $\bar{x}$ be distance of the centre of mass from AB on the axis of symmetry in the direction away from O. Using the formulae for standard uniform bodies:

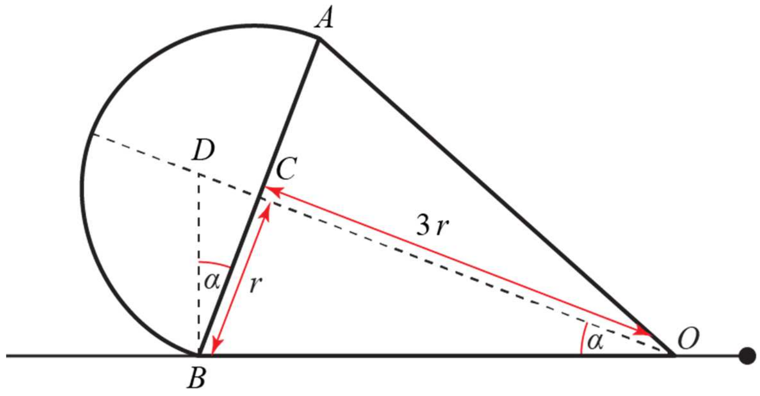
Shape	Mass	Distance of centre of mass from AB
Hemisphere	M	$\frac{3}{8}r$
Cone	m	$-\frac{3}{4}r$
Toy	$m+M$	$\bar{x}$

Taking moments about AB:

$$(m+M)\bar{x}=\frac{3}{8}Mr-\frac{3}{4}mr=\frac{3r}{8}(M-2m)$$

$$\bar{x}=\frac{3(M-2m)}{8(M+m)}r \qquad \text{as required}$$

27 b Let D be the point on the axis of symmetry vertically above B when the toy is placed on a horizontal surface. The toy will not remain in equilibrium if its centre of mass is not on the line segment OD, i.e if $\bar{x} > CD$.



From the diagram: $\tan\alpha = \frac{r}{3r} = \frac{CD}{r} \Rightarrow CD = \frac{1}{3}r$

So $\bar{x} > CD \Rightarrow \frac{3(M-2m)}{8(M+m)}r > \frac{1}{3}r$

$\Rightarrow 9(M-2m) > 8(M+m)$

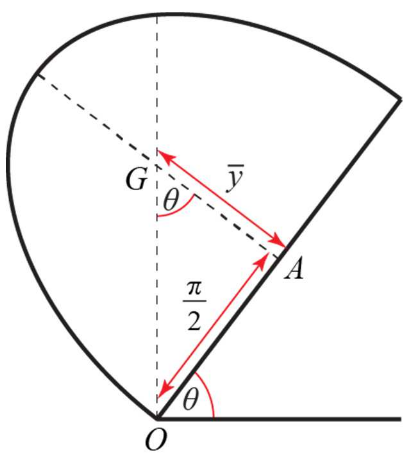
$\Rightarrow M > 26m$ as required

28 a Using $\bar{y} = \dfrac{\int \frac{1}{2}y^2\,dx}{\int y\,dx}$ with $y = \sin x$ gives:

$$\bar{y} = \frac{\int_0^\pi \frac{1}{2}\sin^2 x\,dx}{\int_0^\pi \sin x\,dx} = \frac{\frac{1}{4}\int_0^\pi 1-\cos 2x\,dx}{\int_0^\pi \sin x\,dx}$$

$$= \frac{1}{4}\frac{\left[x - \frac{1}{2}\sin 2x\right]_0^\pi}{\left[-\cos x\right]_0^\pi} = \frac{1}{4}\frac{\pi}{(1+1)} = \frac{\pi}{8}$$

b When S is on the point of toppling, G is above O. Let A be the point midway along the base of S.



$\tan\theta = \frac{\frac{\pi}{2}}{\bar{y}} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$

$\Rightarrow \theta = 76°$ (to the nearest degree)

29 a Let ρ be the mass per unit volume of the material and $\bar{x}$ be the distance of the centre of mass of the solid S from O. Using the formulae for standard uniform bodies:

Shape	Mass	Mass ratios	Distance of centre of mass from O
Cylinder	$\pi\rho(2a)^2\left(\frac{3}{2}a\right)$	6	$\frac{3}{4}a$
Hemisphere	$\frac{2}{3}\pi\rho a^3$	$\frac{2}{3}$	$\frac{3}{8}a$
Solid, S	$\pi\rho\left(6a^3-\frac{2}{3}a^3\right)$	$\frac{16}{3}$	$\bar{x}$

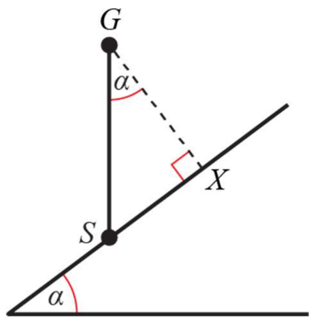
Taking moments about O:

$$\frac{16}{3}\bar{x}=6\times\frac{3}{4}a-\frac{2}{3}\times\frac{3}{8}a$$

$$=\frac{9}{2}a-\frac{1}{4}a=\frac{17}{4}a$$

$$\bar{x}=\frac{51a}{64}=0.797a \text{ (3 s.f.)}$$

b On the point of toppling: G is above S – the lowest point on the bottom circular face.

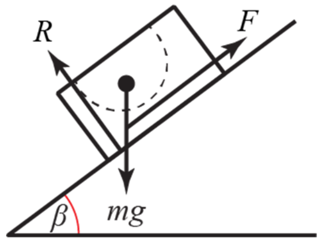


Let X be the centre of the base of the cylinder.

$$\tan\alpha=\frac{SX}{XG}=\frac{2a}{\frac{3}{2}a-\bar{x}}=\frac{64\times 2}{96-51}=\frac{128}{45}$$ substituting value for $\bar{x}$ from part **a**

$\Rightarrow \alpha = 70.6°$ (3 s.f.)

29 c There are three forces acting on S, its weight, the reaction force and friction.



When S is on the point of sliding $F = \mu R = 0.8R$

Resolving perpendicular to the plane

R (↖): $R - mg\cos\beta = 0 \Rightarrow R = mg\cos\beta$

Resolving along the plane

R (↗): $F - mg\sin\beta = 0 \Rightarrow F = mg\sin\beta$

Using $F = 0.8R$ gives

$mg\sin\beta = 0.8mg\cos\beta \Rightarrow \tan\beta = 0.8$

So $\beta = 38.7°$ (3 s.f.)

30 a Using the formulae for standard uniform bodies:

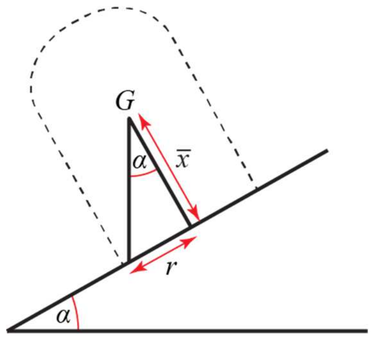
Shape	Mass	Distance of centre of mass from O
Hemisphere, H	$2M$	$h + \frac{3}{8}r$
Cylinder, C	$3M$	$\frac{h}{2}$
Body	$5M$	$\bar{x}$

Taking moments about O:

$$5M\bar{x} = 2M\left(h + \frac{3}{8}r\right) + 3M \times \frac{h}{2} = 2h + \frac{3}{4}r + \frac{3h}{2} = \frac{7h}{2} + \frac{3r}{4}$$

$$\bar{x} = \frac{14h + 3r}{20}$$

30 b When the body is on the point of toppling, its centre of mass G lies directly above a point on the plane which is at the lowest point on its cylindrical base.



From the diagram: $\tan\alpha = \frac{r}{\bar{x}} = \frac{20r}{14h+3r}$

As $\tan\alpha = \frac{4}{3}$, this gives $\frac{20r}{14h+3r} = \frac{4}{3}$

$\Rightarrow 60r = 56h + 12r$

$\Rightarrow 48r = 56h$

$\Rightarrow h = \frac{48}{56}r = \frac{6}{7}r$

31 a Let ρ be the mass per unit volume of the material in the cylinder and $\bar{x}$ be the distance of the centre of mass of the top from O. Using the formulae for standard uniform bodies:

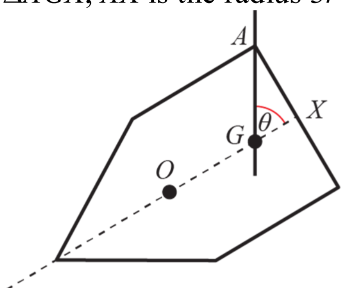
Shape	Mass	Mass ratio	Position of centre of mass from O
Cylinder	$36\pi\rho r^3$	1	$2r$
Cone	$36\pi\rho r^3$	1	$-r$
Toy	$72\pi\rho r^3$	2	$\bar{x}$

Taking moments about O:

$2\bar{x} = 1\times 2r + 1\times(-r) = r$

$\bar{x} = \frac{r}{2}$

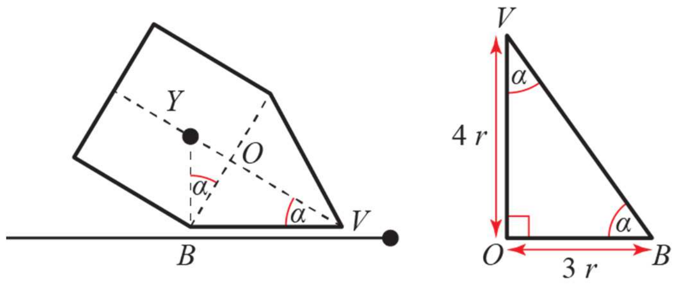
b In ΔAGX, AX is the radius $3r$ of the cylinder and X is the centre of the base of the cylinder.



$\tan\theta = \frac{AX}{GX} = \frac{AX}{OX - OG} = \frac{3r}{4r - \bar{x}} = \frac{3r}{4r - \frac{r}{2}} = \frac{6}{7}$

$\Rightarrow \theta = 47.5°$ (1 d.p.)

31 c The toy will not topple if G (the centre of mass) is vertically above a point between B and V. Let Y be the point directly above B on the axis of symmetry of the toy.



$\tan\alpha = \frac{3r}{4r}$ from ΔVOB and $\tan\alpha = \frac{OY}{OB} = \frac{OY}{3r}$ from ΔBOY

So $\frac{OY}{3r} = \frac{3}{4} \Rightarrow OY = \frac{9r}{4}$

But $OG = \frac{r}{2}$ so $OG < OY$

This means that the centre of mass is above the face of the body in contact with the place, so the toy will not topple.

Challenge

1 The two particles have equal but opposite velocities just before collision at point A. Let their velocities at point A be u and $-u$ respectively.

After collision, P_1 travels three times the distance travelled by P_2 before colliding at point B. Therefore, the velocities after collision are $-3v$ and v respectively.

Using conservation of momentum:

$$mu + 2m(-u) = m(-3v) + 2mv$$

$$\Rightarrow u = v$$

Coefficient of restitution: $e = \frac{u-(-u)}{u-(-3u)} = \frac{1}{2}$

Challenge

2 a Let the instantaneous speed at any point be w and the normal reaction between the ball and the ring is N. Then resolving towards the centre of the circle:

$N = \frac{mw^2}{R}$ using Newton's second law

So the frictional force opposite to the direction of motion is given by:

$$F = \mu \frac{mw^2}{R}$$

Resolving in the direction of motion:

$$m\frac{\mathrm{d}w}{\mathrm{d}t} = -\mu \frac{mw^2}{R}$$

Let v be the velocity at time t, so integrating:

$$\int_u^v \frac{1}{w^2}\,\mathrm{d}w = -\frac{\mu}{R}\int_0^t \mathrm{d}t$$

$$\Rightarrow \left[-\frac{1}{w}\right]_u^v = -\frac{\mu}{R}[t]_0^t$$

$$\Rightarrow -\frac{1}{v} + \frac{1}{u} = -\frac{\mu}{R}t$$

$$\Rightarrow \frac{1}{v} = \frac{1}{u} + \frac{\mu}{R}t = \frac{R + u\mu t}{uR}$$

$$\Rightarrow v = \frac{uR}{R + u\mu t}$$

b From part a $v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{uR}{R + u\mu t}$

The ball completes one revolution in time t when $x = 2\pi R = \pi$

$$\int_0^{\pi} 1\,\mathrm{d}x = \int_0^t \frac{uR}{R + u\mu t}\,\mathrm{d}t == \int_0^t \frac{20}{0.5 + 10t}\,\mathrm{d}t$$

$$\Rightarrow \pi = [2\ln(0.5 + 10t)]_0^t = 2(\ln(0.5 + 10t) - \ln 0.5)$$

$$\Rightarrow \pi = 2\ln(1 + 20t)$$

$$\Rightarrow 20t = e^{\frac{\pi}{2}} - 1$$

$$\Rightarrow t = \frac{e^{\frac{\pi}{2}} - 1}{20} = 0.191\text{s (3 s.f.)}$$